(II) Application to groups with various properties

(II A) Hopfian groups

Def G is Hopfian if every surjective homomorphism $f: G \to G$ is an isomorphism.

Ex: 1) $\mathbb{Z}$ is Hopfian.
2) $\mathbb{F}(n)$ free groups are Hopfian (proof to be given later).

Theorem (Higman) There exist finitely presented groups which are not Hopfian.

Proof.
Consider two copies of the so-called Baumslag-Solitar $BS(1,2)$ group

$$A = \langle a, s \mid s a s^{-1} = a^2 \rangle, \quad B = \langle b, t \mid t b t^{-1} = b^2 \rangle.$$

Observe that $BS(1,2)$ is isomorphic to

$$\mathbb{Z}[\tfrac{1}{2}] \rtimes \mathbb{Z}$$

where $\mathbb{Z}[\frac{1}{2}] = \{ \frac{p}{2^m},\ p, m \in \mathbb{Z} \} \subset \mathbb{Q}$ and

$\mathbb{Z}$ acts on $\mathbb{Z}[\frac{1}{2}]$ by $(1, \frac{p}{2^m}) \longrightarrow \frac{p}{2^{m-1}}$

In particular a is of infinite order in $BS(1,2)$.

Let $\mathbb{Z} \hookrightarrow A$, $\mathbb{Z} \hookrightarrow B$ be given by $1 \longmapsto a$, $1 \longmapsto b$.

Set $G = A \underset{\mathbb{Z}}{*} B$.

Consider $\alpha: A \longrightarrow G$, $\alpha(a) = a^2$, $\alpha(s) = s$; $\beta: B \to G$, $\beta(b) = b^2$, $\beta(t) = t$

Then $\alpha|_{\mathbb{Z} = \langle a \rangle} = \beta|_{\mathbb{Z} = \langle b \rangle}$ and thus it induces

a map $\mu: A \underset{\mathbb{Z}}{*} B = G \longrightarrow G$.

We have $a^2, s, t \in \mu(G) \Rightarrow a = s^{-1} a^2 s \in \mu(G) \Rightarrow$
$\mu(G) = G$. (Because G is generated by $a = b$, s, t)

word (choosing carefully the representative elements for the cosets). Thus

$$g \neq 1 \quad \text{in } G.$$

However

$$\mu(g) = (s^{-1}a^2 s)(t^{-1}b^{-2}t) = a b^{-1} = 1$$

Thus $g \in \ker \mu \neq (1)$. We obtained a surjective homomorphism $\mu : G \to G$ with $\ker \mu$ non-trivial. Therefore G is not Hopfian $\square$.

II B) Constructing free groups

Theorem If A, B are two groups and K is the kernel of the canonical morphism

$$A * B \longrightarrow A \times B$$

then K is a free group, freely generated by the elements

$$X = \{ a b a^{-1} b^{-1}, \quad a \in A - \{1\}, \ b \in B \setminus \{1\} \}.$$

Proof 1) Let L be the subgroup of $A*B$ generated by these commutators from X. If $a' \in A$, $a b a^{-1} b^{-1} \in X$ then

$$a'^{-1}[a,b]\, a' = a'^{-1} a^{-1} b^{-1} a b\, a' = [a a', b][a', b]^{-1} \in L$$

where we set $[a,b] = a^{-1}b^{-1}ab$. Thus

$$a'^{-1} L\, a' = L \qquad \forall a \in A$$

and similarly $b'^{-1} L b' = L \quad \forall b \in B$. Thus L is a normal subgroup of $A*B$. It follows that $L = K$ because K is the smallest normal subgroup for which the quotient $A*B/K$ has all commutators of type $[a,b]$ zero.

2) In order to prove that K is $\mathbb{F}(X)$ the free group on X it suffices to see that any reduced word

$$x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}, \quad \varepsilon_i \in \{\pm 1\}$$ is ~~nonzero~~ nontrivial in K.

Take $x_i = [a_i, b_i]$. We assume this word reduced hence

if $\begin{pmatrix} a_i = a_{i+1} \\ b_i = b_{i+1} \end{pmatrix} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$.

Let $g = [a_1, b_1]^{\varepsilon_1} \dots [a_n, b_n]^{\varepsilon_n} \in A * B$.

By the Van der Waerden structure theorem we can uniquely write g as a product

(*) $g = \alpha_1 \beta_1 \alpha_2 \dots \alpha_N \beta_N$, $\alpha_i \in A$, $\beta_i \in B$.

We call N the length of g, $\ell(g)$

Lemma: i) $\ell(g) \geqslant 3$.

ii) if $\varepsilon_n = 1$ then the word (*) ends with $\underline{a_n b_n}$.

otherwise with $\underline{b_n a_n}$

Proof of lemma:

Clear if $n=1$. Use induction on n. Let us suppose $\varepsilon_{n-1} = 1$, and the reduced product for

$$g' = [a_1, b_1]^{\varepsilon_1} \dots [a_{n-1}, b_{n-1}]^{\varepsilon_{n-1}}$$

reads

$$g' = s_1 \dots s_p \, a_{n-1} b_{n-1}, \quad p \geqslant n.$$

- if $\varepsilon_n = 1 \Rightarrow$

$$g = s_1 \dots s_p \, a_{n-1} b_{n-1} \, a_n^{-1} b_n^{-1} a_n b_n$$

which is a reduced word of length $\geqslant n+6$ ending with $\underline{a_n b_n}$.

- if $\varepsilon_n = -1$ then

$$g = s_1 \dots s_p \, a_{n-1} (b_{n-1} b_n)^{-1} a_n^{-1} b_n a_n$$

– if $b_{n-1} b_n \neq 1$ this is a reduced decomposition length $\geqslant n+5$ ending with $b_n a_n$.

– if $b_{n-1} b_n = 1$ then $a_{n-1} \neq a_n$ (as $\varepsilon_{n-1} = \varepsilon_n = +1$).

thus

$$g = s_1 \dots s_p (a_{n-1} a_n^{-1}) b_n a_n, \text{ of length} \geqslant n+3 \quad \square.$$

II C 2-generators groups are enough

Theorem (Higman-Neumann-Neumann 1949) Every countable group can be embedded into a 2-generator group.

Proof if the group G is finite then it is the subgroup of a symmetric group Σ_n on n elements and this can be generated by 2 elements.

If G is infinite choose an enumeration of its elements

$$G = \{ g_0 = 1, \ g_1, g_2, \dots \}$$

Let $U = \langle u, v \rangle \cong \mathbb{F}(2)$, $B = \langle a, b \rangle \simeq \mathbb{F}(2)$ be two copies of the free group.

- The subgroup $S \subset U$ generated by $s_U = \{ v^n u v^{-n}, n \in \mathbb{Z}_+ \} \subset U$ is a free group in these generators. Moreover $U \hookrightarrow G * U$ and hence $S \subset G * U$ is a free group generated by s_U.
- The same holds true for the subgroup generated by $t_a = \{ b^n a b^{-n}, n \in \mathbb{Z}_+ \} \subset B$; Thus we have a subgroup free $T = \langle t_a \rangle \subset B$.
- The element $\{ g_0 u, \ g_1 vuv^{-1}, \dots, g_n v^n u v^{-n}, \dots \}$ freely generate a subgroup of $G * U$; in fact the projection of any non-empty reduced word in U is a non-trivial element of S. Let then H denote the free subgroup generated by these elements of $G * U$.
- Of course H and T are isomorphic free groups, by setting $\theta : T \to H$, $\theta(b^n a b^{-n}) = g_n v^n u v^{-n}, \forall n$.
- Let consider the HNN extension of $G * U$ associated to θ

$$P = (G * U)*_\theta = \langle G * U, s \mid s t s^{-1} = \theta(t), \forall t \in T \rangle$$

In the HNN extension P we have

$$g_0 u = a, \quad g_1 v u v^{-1} = b a b^{-1}, \dots$$

and so $a = u$ (because $g_0 = 1$) and any g_k can be expressed as a word in a, b, v; in particular

P is generated by a, b, v.

Remark that v and a freely generate a free group of rank 2 and a, b freely generate a free group of rank 2

The isomorphism φ between these subgroups sending

$$v \to a$$
$$a \to b$$

induces a HNN extension

$$E = \left\langle P, w \;\middle|\; \begin{array}{l} w v w^{-1} = a \\ w a w^{-1} = b \end{array} \right\rangle$$

The group E is generated by w and v and contains P which contains $G * U \supseteq G$. Therefore G is contained in a 2-generator group. $\square$

II D) Groups without finite index subgroups (non-trivial)

Proposition (Higman). Let G be the group with presentation

$$\left\langle x_1, x_2, x_3, x_4 \;\middle|\; \begin{array}{l} x_2 x_1 x_2^{-1} = x_1^2 \\ x_3 x_2 x_3^{-1} = x_2^2 \\ x_4 x_3 x_4^{-1} = x_3^2 \\ x_1 x_4 x_1^{-1} = x_4^2 \end{array} \right\rangle$$

a) Any subgroup of finite index in G coincide with G.

b) G is infinite.

Proof a) If $\Gamma \subset G$ is of finite index then it contains a normal subgroup $\Gamma' \triangleleft G$ with finite index. Thus it suffices to show that any finite quotient of G is trivial.

Let $\varphi: G \to \overline{G}$, $\overline{G}$ finite be a morphism surjective.
Assume $\overline{x}_i = \varphi(x_i)$ has order n_i; for some i, $n_i > 1$ otherwise $\overline{G} = 1$.
Let p prime be smallest non-trivial divisor of n_i, say p / n_1.

$$\overline{x}_1^{2^{n_2}} = \overline{x}_2^{n_2} \, \overline{x}_1 \, \overline{x}_2^{-n_2} = \overline{x}_1 \implies 2^{n_2} \equiv 1 \mod n_1$$

and thus $2^{n_2} \equiv 1 \pmod p$. Thus $p \neq 2$. Also $2 \not\equiv 1 \pmod p$.
The order N of 2 in $(\mathbb{Z}/p\mathbb{Z})^*$ verifies then

$$1 < N \leq p-1$$

and the congruence above is equivalent to $n_2 \equiv 0 \pmod N$.
Let p' be a prime factor of N; then $n_2 \equiv 0 \pmod{p'}$
and $p' \leq N \leq p-1$ contradicting the minimality of p. $\square$

b) Let $G_{1,2}$ be the Baumslag-Solitar group $BS(1,2)$
isomorphic to $\langle x_1, x_2 \mid x_2 x_1 x_2^{-1} = x_1^2 \rangle$; it contains
$G_1 = \langle x_1 \rangle$ and $G_2 = \langle x_2 \rangle$ which are isomorphic to $\mathbb{Z}$.
Define similarly G_{23}, G_{34}, G_{41} and their subgroups.
Let then $G_{123} = G_{12} \underset{G_2}{*} G_{23}$, $G_{341} = G_{34} \underset{G_4}{*} G_{41}$.

The subgroup generated by G_1 and G_3 in G_{123} is isomorphic
to $G_1 * G_3 \cong \mathbb{F}(2)$.
In fact, if $H_i \subset G_i$ and $H_i \cap A = (1)$ then the subgroup
generated by $H_1 \cup H_2$ into $G_1 \underset{A}{*} G_2$ is $H_1 * H_2$.
The subgroup generated by G_1 and G_3 in G_{341} is also
isomorphic to $\mathbb{F}(2)$. But now

$$G = G_{123} \underset{\mathbb{F}(2)}{*} G_{341}$$

and as an amalgamated product it is infinite $\square$.

Exercise: The group presented as above but with only 3
generators and 3 cyclic relations is trivial.

III Bruschko Theorem.

Theorem Let $\varphi : G \longrightarrow \underset{\alpha \in I}{*} A_\alpha$ be a homomorphism of a free group G surjective onto the free product of groups A_α. Then there exists a free generating set T of G such that

$$\varphi(T) \subset \bigcup_\alpha A_\alpha$$

Corollary Let the rank of a group be the minimal number of its generators. Then the rank of a free product is the sum of the ranks.

Proof of the theorem (after Stallings, Wagner, Imrich)

III A)

Def: The Cayley graph $\Gamma(G,S)$ of a group G generated by S is the graph with vertex set G and edges

$$\{(g,gs),\ g\in G,\ s\in S\}$$

with endpoints g and gs

Setting g = origin, gs the end point (terminus) of (g,gs) gives edges an orientation.

Observe that the map $\lambda_a : G \to G$ given by $\lambda_a(g) = ag$ induces an ~~left action~~ automorphism $\lambda_a : \Gamma(G,S) \to \Gamma(G,S)$.

Moreover $\lambda_{ab} = \lambda_a \circ \lambda_b$ hence we have a left action of G

$$G \times \Gamma(G,S) \to \Gamma(G,S).$$

which is fixed point free and transitive.

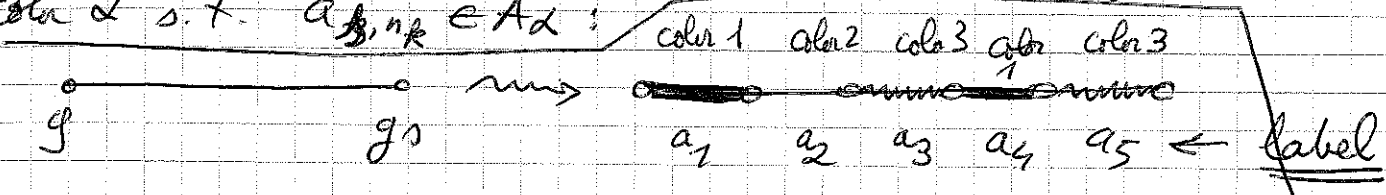
III B) Let G be freely generated by the set S. The Cayley graph $\Gamma(G,S)$ is then a tree.

We can assume that $\varphi(s) \neq 1$, for any $s\in S$. Write then $\varphi(s)$ as a reduced word:

$$\varphi(s) = a_{s,1}\, a_{s,2} \cdots a_{s,n_s}$$

where each $a_{s,i}$ belongs to some $A_{\alpha(s,i)}$.

We divide every edge (g, gs) of $\Gamma(G,S)$ into n_s segments which are labeled $a_{s,1}, \dots, a_{s,n_s}$ and color segment k by color α s.t. $a_{s,n_k} \in A_\alpha$:



Example: $\varphi(s) = a_1\, a_2\, a_3\, a_4\, a_5$, where

$a_1 \in A_1 \quad a_2 \in A_2 \quad a_3 \in A_3 \quad a_4 \in A_1 \quad a_5 \in A_3$

Let then X be the new colored graph. Then λ_a extends to a G-action on X.

III C) Define $\Psi : \{\text{Edges of } X\} \longrightarrow A = * A_\alpha$

$\Psi(\text{edge})$ = label of the edge. This extends to the path-groupoid $\Pi(X) = \{$ paths in X up to homotopy fixing the end points $\}$.

$$\Psi : \Pi(X) \longrightarrow A .$$

Remark also that $\forall g \in G$ and $w \in \Pi(X)$ we have:

$$\Psi(\lambda_g w) = \Psi(w)$$

III D) Let Y be a graph having the same vertices as X and to each x, y vertices we add an uncolored edge joining x to y if there exists a path w in X joining x to y with the property that

$$\Psi(w) = 1 .$$

Observe that G acts freely on Y. Set also $\Psi(e) = 1$ for any uncolored edge. Consequently

$$\Psi(w) = 1 \text{ for any closed path } w \text{ in } Y.$$

III E — Lemma: There exists a maximal spanning forest $F \subset Y$ s.t.

i) F contains only uncolored edges.

ii) F is G-invariant

iii) Let f be a connected component of F. Then two vertices of f are equivalent under the action of G iff they coincide.

iv) Every colored cycle in the graph $Z = Y/F$ (obtained by contracting every component of F to a point) has a monochromatic subcycle.

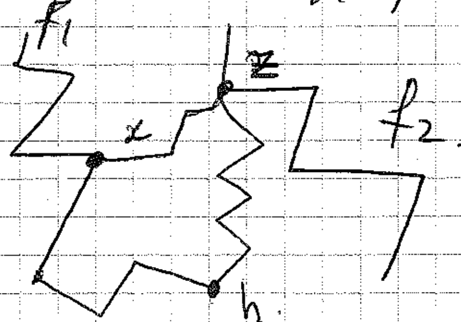
<u>Proof of lemma</u>: Take the forest consisting of the vertices of Y which satisfies all properties; if we have $F_1 \subset F_2 \subset \dots$ satisfying lemma then $\bigcup F_i$ also satisfies requirements of the lemma. Thus Zorn's lemma concludes. $\square$

<u>Rk</u>: F is G-invariant and (iii) $\Rightarrow$ $Z = Y/F$ is freely acted by G.

III F — Lemma: G acts transitively on Z.

<u>Proof of lemma</u>: Suppose the contrary. Thus not all components of F are equivalent under the G-action on Y. Let $f_1 \ni 1 \in \Gamma(G,S)$ and f_2 be two inequivalent components of F. Then consider a colored path x joining f_1 to f in Y. Since φ is surjective there exists $h \in G$ such that

$$\varphi(h) = \varphi(x) \qquad (*)$$



Let endpoint of path x be $z \in f_2$.

Then we have

(**). $\Psi(v) = 1$

for any path v in X joining $\underset{\hat{X}}{h}$ to $\underset{\hat{X}}{z}$, because of (*).

Different elements of G belong to different components of F and thus we can consider them as subset of vertices of Z.

We want to consider a path w satisfying (**) of minimal length i.e. fulfilling:

1. – w is a path in Z containing only colored edges from vertex $g \in G \subset Z$ to another vertex of Z not in the ~~orbit of~~ subset G.
2. – w is of minimal length among such paths
3. – $\Psi(w) = 1$.

Decompose then w as a product of maximal monochromatic subpaths

$$w = w_1 w_2 \dots w_n$$

Thus $\Psi(w_1)\Psi(w_2)\dots\Psi(w_n) = 1$ in A.

The structure theorem implies that for some j

$$\Psi(w_j) = 1.$$

The path w_j is monochromatic and its endpoints are not G-equivalent; otherwise we can replace w by

$$w' = w_1 w_2 \dots w_{j-1} \ \lambda_g(w_{j+1} \dots w_n)$$

for some g. This w' also satisfies (1–3) but it is shorter than w, contradicting our choice.

Therefore the components of F mapped to endpoints of w_j are also NOT G-equivalent and connected by an uncolored edge because $\Psi(w_j) = 1$. Let e be the uncolored edge.

Contraction of $\lambda g e$ in Z produces only monochromatic colored cycles. Thus

$$F \cup \{\lambda g e,\ g \in G\}$$

contradicts the maximality of F. $\square$

III G We can assume G acts transitively on Z hence vertices of Z are G. Set

$G_\alpha = \{$vertices of Z consisting of 1 and all vertices accessible by a monochromatic path colored $\alpha\}$

Then $G_\alpha \subset G$ is a subgroup.

Lemma: $\boxed{G = \ast_\alpha G_\alpha}$

Proof: G_α generate G obviously. ~~Let~~ $g_1 g_2 \cdots g_n$ with g_i, g_{i+1} in different G_α for $1 \le i \le n$; If

$$g_1 \cdots g_n = 1$$

choose w_1 monochromatic path shortest from 1 to g_1, w_2 from g_1 to $g_1 g_2, \ldots$ w_k from $g_1 \cdots g_{k-1}$ to $g_1 \cdots g_k$. Then $w_1 \cdots w_n$ is a closed path. Therefore it should contain a monochromatic cycle, which is impossible by the choice of w_i. $\square$

Then take T_α generators for G_α and $T = \bigcup T_\alpha$. $\square$

IV More constructions

IVA **Def** 1) The word $w = x_1 x_2 \dots x_n$ is strictly alternating if consecutive terms x_j, x_{j+1} belong to different factors A, B of the amalgamated free product $A *_H B$, $H \hookrightarrow A$, $H \hookrightarrow B$.

2) w is cyclically reduced if $n = 1$ or $n > 1$ and x_1, x_n come from different factors A, B.

Proposition Every word w cyclically reduced of length $n \geq 2$ is an element of infinite order in $A *_H B$.

Proof $w^k = x_1 x_2 \dots x_n x_1 \dots x_n \dots x_n$ is ~~reduced since~~ strictly alternating thus of length $kn \geq 2$ and thus $w^k \neq 1$ in $A *_H B$. $\square$

Theorem (B.H. Neumann) There exists ~~a~~ continuously many non-isomorphic groups generated by 2 elements.

Proof Let p_i denote the i-th prime number and to each increasing sequence $\underline{n} = (n_k)$ associate the group

$$A_{\underline{n}} = \bigoplus_{i=1}^{\infty} \mathbb{Z}/p_{n_i}\mathbb{Z}$$

Observe that $A_{\underline{n}} \simeq A_{\underline{m}}$ iff $\underline{n} = \underline{m}$, since they have the same finite subgroups.

(IIC) Theorem (H-N-N) shows how to embed $A_{\underline{n}} \hookrightarrow G_{\underline{n}}$ a group with 2 generators.

Proposition above shows that in an amalgamated product the elements of finite order are conjugate to an element of one factor. Thus elements of finite order of $G_{\underline{n}}$ are those conjugate to elements of $A_{\underline{n}}$. In particular $G_{\underline{n}} \simeq G_{\underline{m}}$ iff $\underline{n} = \underline{m}$. $\square$

IVB.

Theorem The Baumslag-Solitar group $BS(2,3) = \langle a, s \mid s^{-1} a^2 s = a^3 \rangle$ is not Hopfian.

Proof Let $\varphi : BS(2,3) \longrightarrow BS(2,3)$ given by

$\varphi(a) = a^2$, $\varphi(s) = s$.

We have $a^2, s \in \varphi(G)$ and so also $a \in \varphi(G)$ thus $\varphi(G) = G$.
Moreover the element

$$g = (a^{-1}s^{-1}as)^2 a^{-1} = a^{-1}s^{-1}asa^{-1}s^{-1}asa^{-1}$$

satisfies condition of Britton's lemma. Thus $g \neq 1$ in the HNN extension of $\mathbb{Z}$ (generated by a), namely $BS(2,3)$.
However

$$\varphi(g) = (a^{-2}s^{-1}a^2s)^2 a^{-2} = a^2 a^{-2} = 1.$$

Thus φ surjective and not an isomorphism $\square$.

<u>Remark</u>: Say p, q are meshed if one divides the other or the set of prime divisors is the same. One proves that the Baumslag-Solitar group

$$BS(p,q) = \langle a, s \mid s^{-1}a^p s = a^q \rangle$$

is non Hopfian if p, q are not meshed.

<u>Proof</u> Let d be a prime dividing p and not q. The map

$\varphi(a) = a^d$, $\varphi(s) = s$ defines a homomorphism.

$g = [a^{p/d}, s]^d a^{p-q} = a^{-p/d}s^{-1}a^{p/d}s\,a^{-p/d}s^{-1} \dots s\,a^{p-q}$

is non trivial by Britton's lemma.

However

$$\varphi(g) = [a^p, s]^p a^{d(p-q)} = 1$$

Thus kernel is non-trivial